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NUMERICAL SOLUTION FOR TRANSIENT WELL FLOW IN AN UNCONFINED-CONFINED AQUIFER SYSTEM

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It is quite common that the extraction of groundwater is from more than one aquifer system, where a combination of unconfined and confined aquifers is found. The problem lies in how to find the discharge from each of the aquifers, whether or not it is unconfined or confined in nature. The governing equations, which are second order partial differential equations, are solved numerically using a finite difference method. To obtain a solution, computer code has been written in MATLAB, and a program has been developed. The layered aquifer is idealized as an unconfined-confined aquifer system including the well in which the solution domain is stated to be two-dimensional. This method has yielded the relationship between drawdown and time in the well, and is also able to determine the discharge from each of the aquifers, whether unconfined or confined.

INTRODUCTION

A common problem in development of groundwater resources is the quantity and the quality of the water. Uncontrolled groundwater exploitation can lower the water table or, by decreasing pressure heads of groundwater, can cause the contamination of groundwater by toxic waste from dump sites and intrusion of seawater (Mohd. Kassim et al., 1995). It is often found in practice that the exploitation of groundwater is from more than one aquifer, especially from an unconfined-confined aquifer system. It is desirable to find an analytical or numerical solution of the flow mechanism of this aquifer system.

Abdul Khader and Veerankutty (1975) solved the flow mechanism in the unconfined-confined aquifer system analytically by using the Laplace Transformation Method. However, the application of their solution is limited and not easily used. The solution requires values of the following parameters: transmisivity of each aquifer, specific yield of the unconfined aquifer, storativity of the confined aquifer, well diameter, well drawdown, well discharge and water level from available graphs.

NUMERICAL MODEL AND SOLUTIONS

The basic equations governing the system are subjected to appropriate initial and boundary conditions using the following assumptions (Figure 1):

- For the boundary condition describing the discharge, the well is idealized as a sink. That is, instead of taking the actual cylindrical section for water entry, a hypothetical line is assumed. This simplification is frequently used in treating problems where the well radius is very small. Equations of flow towards wells in unconfined aquifers and confined aquifers, partially penetrating wells and multi-aquifer wells, have all been derived under this assumption.
- In applying the boundary condition at the free surface in an unconfined aquifer, the lowering of the water table is assumed to be same as changes in head along the static water table position where radial flow components are neglected.

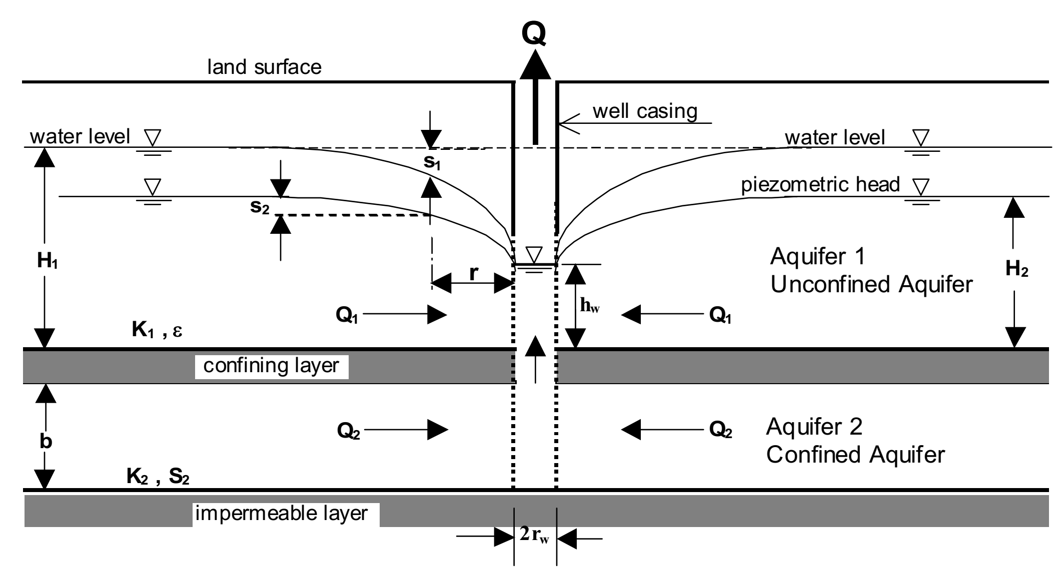

Figure 1. Definition sketch of unconfined-confined aquifer system.

- In the case of the unconfined aquifer, the specific storage is assumed to be small compared to the specific yield. It should be noted that compressibility may occur in the early periods of flow, and it may not be completely ignored (Lohman, 1972). Nevertheless, in the present analysis, the contribution from aquifer compression in the unconfined aquifer is neglected.

- In applying the boundary condition describing drawdown at the well face, the well losses are neglected. The head at the well face in any aquifer at any time (t > 0) is assumed to be equal to that in the well and identical to that of the aquifers. Equations for drawdown around a multi-aquifer well were derived using this assumption.

- The aquifers are assumed to be separated by completely impermeable layers and are connected only through the well. Therefore leakage of any kind is not considered in the analysis.

With the above assumptions, the governing equations, initial value and boundary value problem can be set up as follows:

Aquifer 1 (Unconfined):

$$\frac{\partial^2 s_1}{\partial r^2} + \frac{1}{r}\frac{\partial s_1}{\partial r} + \frac{\partial^2 s_1}{\partial z^2} = 0 \tag{1}$$

Boundary condition at $z = H_1$:

$$\frac{\partial s_1}{\partial z} + \frac{\varepsilon}{K_1}\frac{\partial s_1}{\partial t} = 0 \tag{2}$$

Boundary condition at $z = 0$:

$$\frac{\partial s_1}{\partial z} = 0 \tag{3}$$

Boundary condition at $r = 0$:

$$\lim_{r \to 0}\left\{2\pi T_1 \frac{\partial s_1}{\partial r}\right\} = -Q_1(t) \tag{4}$$

Boundary condition at $r = \infty$:

$$s_1(\infty, z, t) = 0 \tag{5}$$

Initial condition:

$$s_1(r, z, 0) = 0 \tag{6}$$

Aquifer 2 (Confined):

$$\frac{\partial^2 s_2}{\partial r^2} + \frac{1}{r}\frac{\partial s_2}{\partial r} = \frac{1}{v}\frac{\partial s_2}{\partial t} \tag{7}$$

Boundary condition at $r = 0$:

$$\lim_{r \to 0}\left\{2\pi T_2 \frac{\partial s_2}{\partial r}\right\} = -Q_2(t) \tag{8}$$

Boundary condition at $r = \infty$:

$$s_2(\infty, t) = 0 \tag{9}$$

Initial value:

$$s_2(r, 0) = 0 \tag{10}$$

Equations at the well:

$$H_1 - s_1(r, h_w, t) = H_2 - s_2(r_w, t) \tag{11}$$

$$Q_1(t) + Q_2(t) = Q \tag{12}$$

where:

s_1, s_2	=	drawdown in each aquifer at any distance, r, from the center of the well, time, t, and height, z, measured from the bottom of the unconfined aquifer [L];
H_1, H_2	=	initial heads in each aquifer [L];
K_1, K_2	=	hydraulic conductivity of aquifer 1 and 2 respectively [L^2T^{-1}];
ε	=	specific yield of aquifer 1;
T_1	=	K_1H_1 = transmisivity of aquifer 1 [L];
T_2	=	K_2b = transmisivity of aquifer 2 [L];
b	=	thickness of aquifer 2 [L];
v	=	T_2/S_2 = hydraulic diffusivity of aquifer 2 [LT^{-1}];
S_2	=	storage coefficient of aquifer 2;
r_w	=	effective radius of the well [L];
h_w	=	water level in the well measured from the bottom of aquifer 1 at any time t [L];
$Q_1(t)$, $Q_2(t)$	=	discharge contributions of aquifer 1 and 2 respectively at any time [L^3T^{-1}].

Discretization of the Unconfined Aquifer

Equations (1) and (2), the governing and the boundary condition equations of aquifer 1 (the unconfined aquifer) respectively are then approximated by spatial discretization using a finite difference formulation. If the domain of the equation is $(x,y) = (0,P)$, $(0,L)$ and the width of each discretization of grid is $h = P/N$ and $k = L/M$, then the formula is expressed as (Anderson, 1984).

$$\left(\frac{\partial^2 s_1}{\partial r^2}\right)_{i,j} = \frac{s_{1(i+1,j)} - 2s_{1(i,j)} + s_{1(i-1,j)}}{h^2} \tag{13}$$

$$\frac{1}{r}\left(\frac{\partial s_1}{\partial r}\right)_{i,j} = \frac{1}{ih}\frac{s_{1(i+1,j)} - s_{1(i-1,j)}}{2h} \tag{14}$$

$$\left(\frac{\partial^2 s_1}{\partial z^2}\right)_{i,j} = \frac{s_{1(i,j+1)} - 2s_{1(i,j)} + s_{1(i,j-1)}}{k^2} \tag{15}$$

$$\left(\frac{\partial s_1}{\partial z}\right)_{i,j} = \frac{\overset{n+1}{S}_{1(i,N+1)} - \overset{n+1}{S}_{1(i,N-1)}}{2k} \tag{16}$$

$$\frac{\varepsilon}{K_1}\left(\frac{\partial s_1}{\partial t}\right)_{i,j} = \frac{\varepsilon}{K_1}\frac{\overset{n+1}{S}_{1(i,N)} - \overset{n}{S}_{1(i,N)}}{u} \tag{17}$$

If $X = s_1$ = the drawdown at aquifer 1, then the discretization of Equations (1) and (2) yields:

$$\frac{X_{i+1,j} - 2X_{i,j} + X_{i-1,j}}{h^2} + \frac{1}{ih}\frac{X_{i+1,j} - X_{i-1,j}}{2h} + \frac{X_{i,j+1} - 2X_{i,j} + X_{i,j-1}}{u^2} = 0 \Leftrightarrow$$

$$\frac{1}{h^2}\left(1 - \frac{1}{2i}\right)X_{i-1,j} + \frac{1}{u^2}X_{i,j-1} - 2\left(\frac{1}{h^2} + \frac{1}{u^2}\right)X_{i,j} + \frac{1}{u^2}X_{i,j+1} + \frac{1}{h^2}\left(1 + \frac{1}{2i}\right)X_{i+1,j} = 0 \tag{18}$$

$$\frac{\overset{n+1}{X}_{i,N+1} - \overset{n+1}{X}_{i,N-1}}{2u} + \frac{\varepsilon}{K_1}\frac{\overset{n+1}{X}_{i,N} - \overset{n}{X}_{i,N}}{k} = 0 \Leftrightarrow$$

$$\overset{n+1}{X}_{i,N+1} = \overset{n+1}{X}_{i,N-1} - \left(\frac{2\varepsilon u}{K_1 k}\right)\overset{n+1}{X}_{i,N} + \left(\frac{2\varepsilon u}{K_1 k}\right)\overset{n}{X}_{i,N} = 0 \quad \text{(required when } j = N) \tag{19}$$

$$X_{i,0} = X_{i,2} \quad \text{(required when } j = 1) \tag{20}$$

$$X_{0,j} = X_{2,j} + \frac{Q_1(t)}{\pi T_1} \quad \text{(required when } i = 1) \tag{21}$$

$$X_{M+1,j} = 0 \quad \text{(required when } i = M) \tag{22}$$

$$\overset{0}{X}_{i,j} = 0 \tag{23}$$

If : $\alpha = \frac{1}{h^2}$, $\beta = \frac{1}{u^2}$, $\gamma = -2(\alpha + \beta)$, $\delta = \left(\frac{2\varepsilon u}{K_1 k}\right)$, $\eta = \frac{\alpha}{2\pi T_1}$, then the matrix equation of unconfined aquifer is obtained as

$$
\begin{bmatrix}
\gamma & 2\beta & 0 & 2\alpha & & & & & \\
\beta & \gamma & \beta & 0 & 2\alpha & & & & \\
0 & 2\beta & \gamma & 0 & 0 & 2\alpha & & & \\
\frac{3}{4}\alpha & 0 & 0 & \gamma & 2\beta & 0 & \frac{5}{4}\alpha & & \\
 & \frac{3}{4}\alpha & 0 & \beta & \gamma & \beta & 0 & \frac{5}{4}\alpha & \\
 & & \frac{3}{4}\alpha & 0 & 2\beta & \gamma & 0 & 0 & \frac{5}{4}\alpha \\
 & & & \frac{5}{6}\alpha & 0 & 0 & \gamma & 2\beta & 0 \\
 & & & & \frac{5}{6}\alpha & 0 & \beta & \gamma & \beta \\
 & & & & & \frac{5}{6}\alpha & 0 & 2\beta & \gamma
\end{bmatrix}
\begin{bmatrix}
X_{11}^{n+1} \\ X_{12}^{n+1} \\ X_{13}^{n+1} \\ X_{21}^{n+1} \\ X_{22}^{n+1} \\ X_{23}^{n+1} \\ X_{31}^{n+1} \\ X_{32}^{n+1} \\ X_{33}^{n+1}
\end{bmatrix}
= -
\begin{bmatrix}
\eta Q_1(t) \\ \eta Q_1(t) \\ \eta Q_1(t) + \delta X_{13}^{n} \\ 0 \\ 0 \\ \delta X_{23}^{n} \\ 0 \\ 0 \\ \delta X_{33}^{n}
\end{bmatrix}
$$

Discretization of the Confined Aquifer

Equations (7) to (9), the governing and the boundary equations of aquifer 2 (the confined aquifer) respectively are then approximated by spatial discretization using a finite difference formulation. If the domain of equation is $(x,y) = (0,P), (0,L)$ and the width of each discretization of grid is $h = P/N$ and $k = L/M$, then the formula is expressed as (Anderson, 1984).

$$\left(\frac{\partial^2 s_2}{\partial r^2}\right)_{i,j} = \frac{s_{2(i+1,j)} - 2s_{2(i,j)} + s_{2(i-1,j)}}{h^2} \tag{25}$$

$$\frac{1}{r}\left(\frac{\partial s_2}{\partial r}\right)_{i,j} = \frac{1}{ih}\frac{s_{2(i+1,j)} - s_{2(i-1,j)}}{2h} \tag{26}$$

$$\frac{1}{v}\left(\frac{\partial s_2}{\partial t}\right)_{i,j} = \frac{1}{v}\frac{s_{2(i,j+1)} - s_{2(i,j-1)}}{k} \tag{27}$$

If $Y = s_2$ = the draw-down at the aquifer 2, then the discretization of Equations (7) to (9) yields:

$$
\begin{aligned}
&\frac{Y_{i+1}^{n+1} - 2Y_i^{n+1} + Y_{i-1}^{n+1}}{h^2} + \frac{1}{ih}\frac{Y_{i+1}^{n+1} - Y_{i-1}^{n+1}}{2h} = \frac{1}{v}\frac{Y_i^{n+1} - Y_i^{n}}{k} \Leftrightarrow \\
&\frac{1}{h^2}\left(1 - \frac{1}{2i}\right)Y_{i-1}^{n+1} - \left(\frac{2}{h^2} + \frac{1}{vk}\right)Y_i^{n+1} + \frac{1}{h^2}\left(1 + \frac{1}{2i}\right)Y_{i+1}^{n+1} = -\frac{1}{vk}Y_i^{n}
\end{aligned} \tag{28}
$$

$$Y_{M+1} = 0 \ \text{(required when } i = M\text{)} \tag{29}$$

$$Y_0 = Y_2 + \frac{Q_2(t)}{\pi T_2} \ \text{(required when } i = 1\text{)} \tag{30}$$

If : $\alpha = \frac{1}{h^2}$, $\rho = \frac{1}{vk}$, $\sigma = -\left(\frac{2}{h^2} + \frac{1}{vk}\right)$, and $\eta = \frac{\alpha}{2\pi T^2}$, then the matrix equation of the confined aquifer is obtained as:

$$\begin{bmatrix} \sigma & 2\alpha & 0 \\ \frac{3}{4}\alpha & \sigma & \frac{5}{4}\alpha \\ 0 & \frac{5}{6}\alpha & \sigma \end{bmatrix} \begin{bmatrix} Y_1^{n+1} \\ Y_2^{n+1} \\ Y_3^{n+1} \end{bmatrix} = - \begin{bmatrix} \rho Y_1^n + \eta Q_2(t) \\ \rho Y_2^n \\ \rho Y_3^n \end{bmatrix} \tag{31}$$

Computer Program Using MATLAB

Equations (24) and (31) are solved in order to obtain the solution vectors X and Y. Components X_{11} and Y_1 are used for solving Equation (11) in order to represent:

$$\Delta = \left| H_1 - X_{11} - H_2 + Y_1 \right| \tag{32}$$

Generally, the algorithm for obtaining solution vectors X and Y is as follows:

[Q_1, Q_2, X, Y] = function of (Q, H_1, H_2)

1. $Q_1 = 0.1$ x Q x $Q_2 = Q - Q_1$
2. To solve Equation (24) for X
3. To solve Equation (31) for Y
4. $\Delta = \left| H_1 - X_{11} - H_2 + Y_1 \right|$
5. If $\Delta > 0.1$, then

   begin

      $Q_1 = 1.1$ x Q

      renew the right hand side of Equation (24)

      $Q_2 = Q - Q_1$

      renew the right hand side of Equation (31)

      go to 2

   end

6. end.

where: $h = P/N$ = width of grid

$k = L/M$ = length of grid

$u = Dt$ = time interval

L = length of the domain

M = order of discretisation of r direction

N = order of discretisation of z direction

P = width of the domain

Δ = a small value close to zero to replace zero value

RESULTS AND DISCUSSION

Transient drawdown at the well obtained from: (1) running the program, (2) physical model data collection, (3) calculation using other relevant theory (Abdul Khader and Veerankutty, 1975), and (4) field data collection are well plotted in Figures 2 and 3. The results show good agreement.

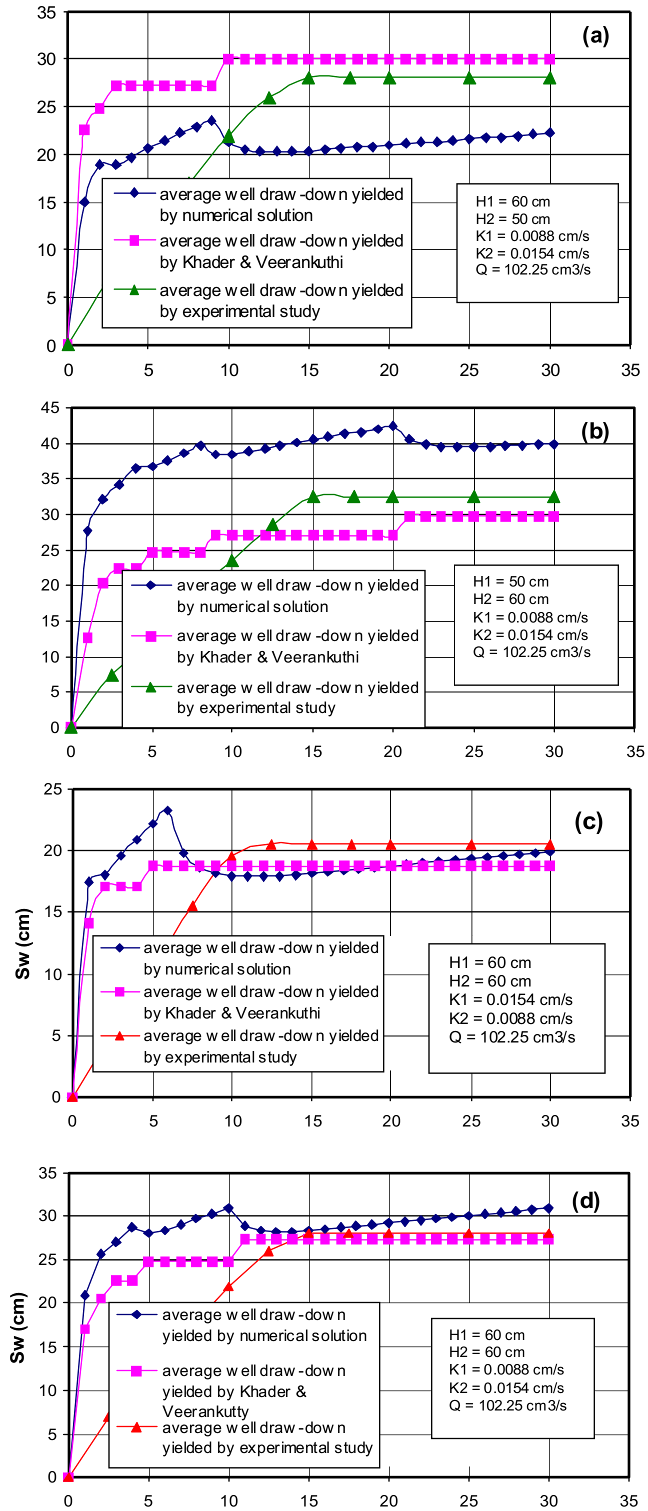

Figure 2. Well draw-down curve, comparison among numerical solution, experimental study and analytical work.

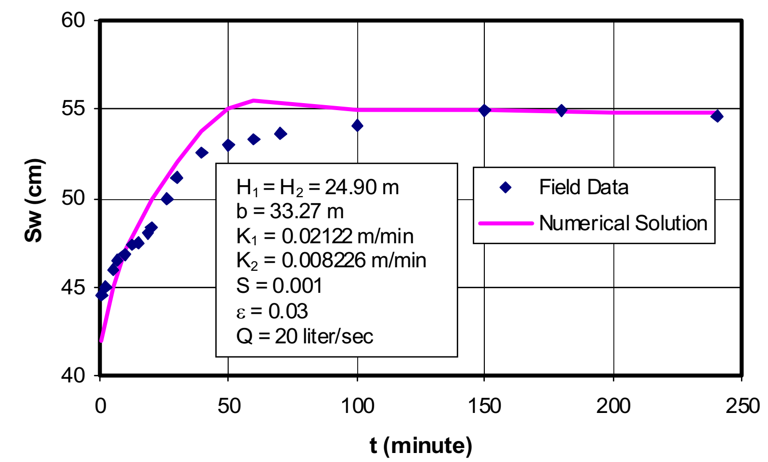

Figure 3. Well drawdown curve, comparison between field data and numerical solution.

It is observed that a steady state is attained after 30-60 seconds. The discrepancies noticed in the graphs are due to the limitation imposed by the bounded nature of the aquifers in the physical model. The discrepancy may also in part caused by the improper input value of the well(h_w), as max (H_1, H_2) must be an even and multiple number of h_w.

The ratio of Q_1 to Q as the initial value is varied to check the resulting values of Q_1, Q_2, s_{w1} and s_{w2}. Various values of H_2, H_1 and h_w are also tested to observe the change in the values of Q_1, Q_2, s_{w1} and s_{w2}. The results of the simulation are listed in Table 1.

Table 1. Results from Running the Program

No.	H_1 cm	H_2 cm	K_1 cm/sec	K_2 cm/sec	Q cm^3/sec	Q_1 cm^3/sec	Q_2 cm^3/sec	s_{w1} cm	s_{w2} cm
1	60.00	60.00	0.0088	0.0154	80.00	33.42	46.58	46.23	50.78
	60.00	60.00	0.0088	0.0154	80.00	31.12	48.88	43.05	51.79
	60.00	60.00	0.0088	0.0154	80.00	31.18	48.82	43.13	51.71
2	60.00	60.00	0.0154	0.0088	80.00	53.82	26.18	42.28	45.48
	60.00	60.00	0.0154	0.0088	80.00	55.24	24.76	43.40	45.81
	60.00	60.00	0.0154	0.0088	80.00	51.45	28.55	40.42	48.03
	60.00	60.00	0.0154	0.0088	80.00	51.54	28.46	40.49	47.85
3	60.00	60.00	0.0088	0.0154	80.00	33.42	46.58	46.23	50.78
	60.00	55.00	0.0088	0.0154	80.00	33.42	46.58	46.23	49.42
	60.00	50.00	0.0088	0.0154	80.00	36.76	43.24	50.85	46.50
	60.00	45.00	0.0088	0.0154	80.00	36.76	43.24	50.85	45.64
4	60.00	60.00	0.0154	0.0088	80.00	53.82	26.18	42.28	45.48
	60.00	55.00	0.0154	0.0088	80.00	53.82	26.18	42.28	44.22
	60.00	50.00	0.0154	0.0088	80.00	59.20	20.80	46.51	37.95
	60.00	45.00	0.0154	0.0088	80.00	59.20	20.80	46.51	35.63
5	60.00	60.00	0.0088	0.0154	80.00	33.42	46.58	46.23	50.78
	55.00	60.00	0.0088	0.0154	80.00	27.62	52.38	41.68	55.39
	50.00	60.00	0.0088	0.0154	80.00	25.11	54.89	41.68	58.33
	45.00	60.00	0.0088	0.0154	80.00	22.82	57.18	42.10	61.11
6	60.00	60.00	0.0154	0.0088	80.00	53.82	26.18	42.28	45.48
	55.00	60.00	0.0154	0.0088	80.00	48.93	31.07	41.93	52.64
	50.00	60.00	0.0154	0.0088	80.00	44.48	35.52	41.93	59.56
	45.00	60.00	0.0154	0.0088	80.00	44.48	35.52	46.59	63.97

The simulation results in Table 1 illustrate that H_2 is sensitive to Q_2 and s_{w2}, provided the given value of K_2 is smaller than K_1. On the other hand, H_1 is sensitive to Q_1, Q_2 and s_{w2} regardless of the ratio of K_1 to K_2. The most sensitive parameter is h_w since h_w changes noticeably due to small alteration of both Q and s_w. However, if it is compared with the results of laboratory experiments, a reasonable well simulation of the model is obtained with h_w = 0.5 x(H_1, H_2). In the case of different initial heads, the suggested value of h_w is 0.4 x(H_1, H_2).

CONCLUSIONS

- The problem of unsteady flow into a well screened to a water table aquifer and a lower confined aquifer is formulated and solved with the help of a numerical technique.
- A type of computer program written in MATLAB has been developed which can be employed to analyze the groundwater flow into a well in an unconfined-confined aquifer system.
- The well discharge and the well drawdown of each aquifer, whether unconfined or confined, can be calculated by the program.
- A two-dimensional solution is more easily solved numerically as the solution enables the representation of the actual field problem by using a complex boundary system.

REFERENCES

Abdul Khader, M.H. and Veerankutty; (1975). Transient Well Flow in An Unconfined-Confined Aquifer System, J. Hydrology, vol.3, 26-30 pp.

Anderson, D.A., J.C. Tannehil and R.H. Pletcher; (1984). Computational Fluid Mechanics and Head Transport. Hemisphere Publ. Corp., New York.

Anderson, M.P. and W.W. Woessner; (1992). Applied Groundwater Modeling. Academic Press, San Diego, California.

Lohman, S.W.; (1972). Groundwater Hydraulics. U.S. Geological Survey Prof. Paper 708.

Mohd. Kassim, A.M., M.H. Abdul Khader and D. Tjahjanto; (1995). Numerical Modeling of Groundwater Flow, A Case Study for Surabaya Aquifer. J. Civil Engineering, UTM, Johor Bahru, vol.8, No.1, 49-62 pp.

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